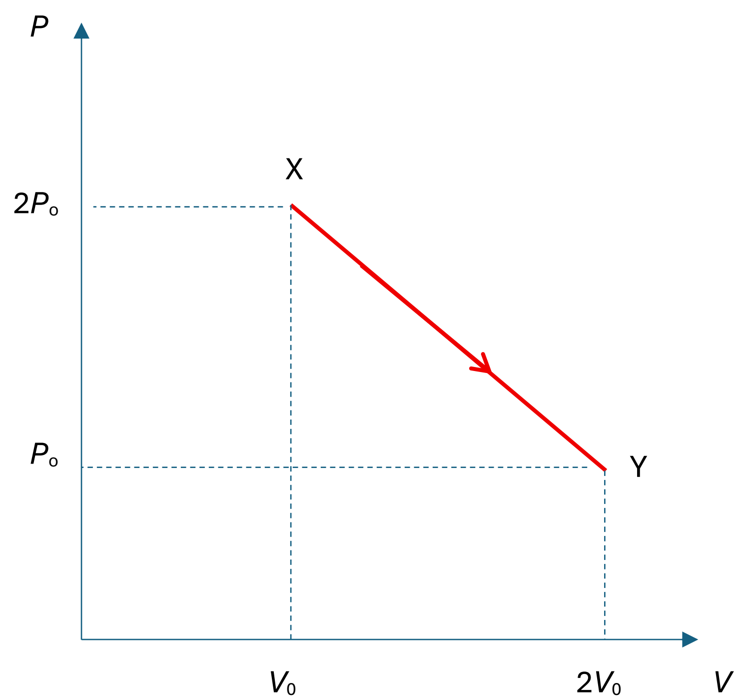
Teacher notes

Topic B

A very subtle point in the calculation of heat entering a system pointed out by M. Jarocki

In the textbook part of a problem (Question 33 in Thermodynamics) involved calculating the heat that enters an ideal gas as the state is changed from X to Y along the straight line shown.



The naïve calculation using $Q = \Delta U + W$ gives:

$\Delta U = 0$ because X and Y have the same temperature and so

$Q = W = \frac{2P_0 + P_0}{2} \times V_0 = \frac{3}{2} P_0 V_0$. If we take for simplicity, $P_0 = V_0 = 1$ then $Q = \frac{3}{2}$ in arbitrary units.

Maciej Jarocki has pointed out a very subtle point that makes this naïve calculation incorrect.

$$dQ = dU + dW = \frac{3}{2}RndT + PdV$$

From $PV = RnT \Rightarrow T = \frac{PV}{Rn} \Rightarrow dT = \frac{dP\,V + PdV}{Rn}$ and so

$$dQ = \frac{3}{2}Rn(\frac{dP\,V + PdV}{Rn}) + PdV = \frac{5}{2}PdV + \frac{3}{2}VdP$$

Along XY, $P = 3P_0 - \frac{P_0}{V_0}V$ and so $dP = -\frac{P_0}{V_0}dV$ and so finally

$$\begin{aligned} dQ &= \frac{5}{2}(3P_0 - \frac{P_0}{V_0}V)\,dV + \frac{3}{2}V(-\frac{P_0}{V_0})\,dV \\ &= (\frac{15}{2}P_0 - \frac{5}{2}\frac{P_0}{V_0}V - \frac{3}{2}\frac{P_0}{V_0}V)\,dV \\ &= (\frac{15}{2}P_0 - 4\frac{P_0}{V_0}V)\,dV \end{aligned}$$

For simplicity take $P_0 = V_0 = 1$ so that $dQ = (\frac{15}{2} - 4V)\,dV$.

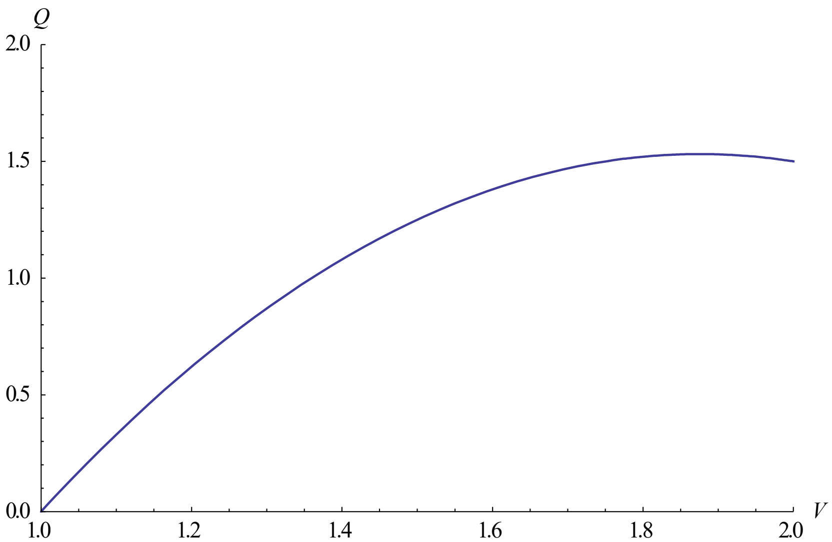
Then

$$\begin{aligned} Q &= \int_1^V (\frac{15}{2} - 4V)\,dV \\ &= (\frac{15}{2}V - 2V^2)\Big|_1^V = (\frac{15}{2}V - 2V^2) - (\frac{15}{2} - 2) \end{aligned}$$

$$Q = \frac{1}{2}(15V - 4V^2 - 11)$$

Substituting $V = 2$ gives $Q = \frac{1}{2}(15 \times 2 - 4 \times 2^2 - 11) = \frac{3}{2}$, the naïve result.

However, plotting Q vs V gives

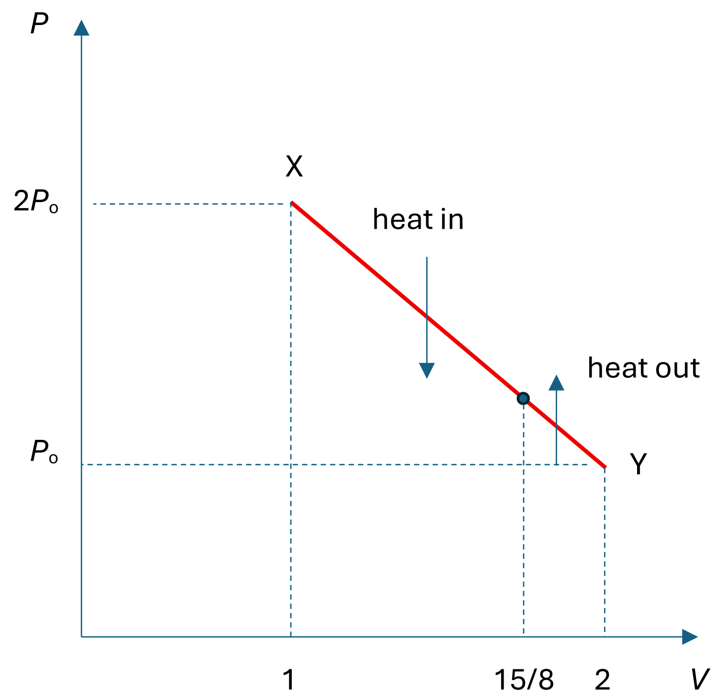


There is a maximum when $\frac{dQ}{dV}=\frac{1}{2}(15-8V)=0 \Rightarrow V=\frac{15}{8}$. Heat decreases after this volume value. This means that heat enters the system from $V = 1$ to $V=\frac{15}{8}$ and **leaves** the system from $V=\frac{15}{8}$ to $V = 2$. The heat that enters is

$$Q=\frac{1}{2}(15\times\frac{15}{8}-4\times(\frac{15}{8})^2-11)=\frac{49}{32}$$

The heat that leaves is

$$Q=\frac{3}{2}-\frac{49}{32}=-\frac{1}{32}.$$



During the expansion of the gas from X to Y the temperature is given by

$$T = \frac{PV}{Rn} = \frac{(3-V)V}{Rn}$$

The maximum temperature is reached when the volume is

$$\frac{dT}{dV} = \frac{(3-2V)}{Rn} = 0 \Rightarrow V = \frac{3}{2}$$

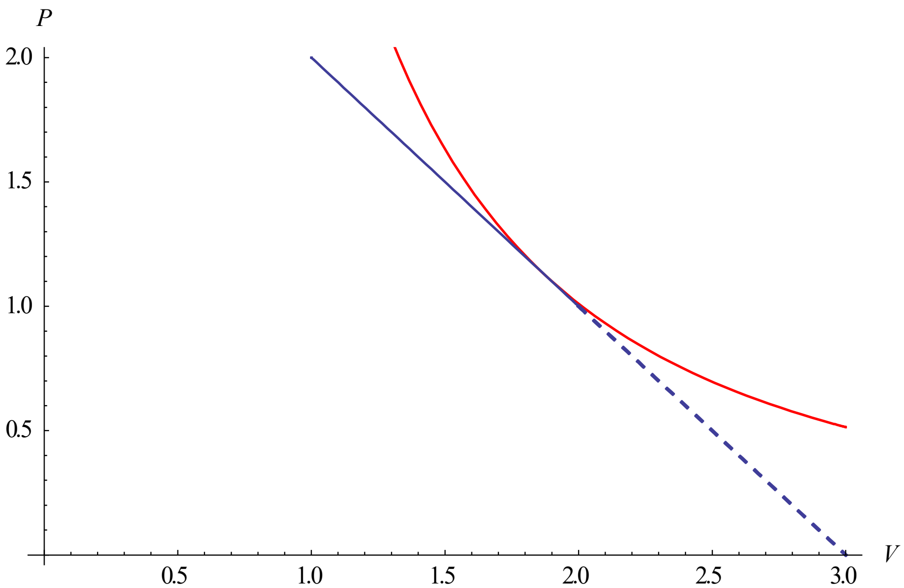
So, from this volume up to $V = 2$ the temperature and hence the internal energy is decreasing. Intuitively we can understand the significance of $V = \frac{15}{8}$ as the volume beyond which the internal energy is decreasing faster than the work done by the gas and so heat leaves the gas.

Another feature of $V = \frac{15}{8}$ is that at this volume the line showing the variation of pressure with volume is tangent to the adiabatic curve at that point as shown below.

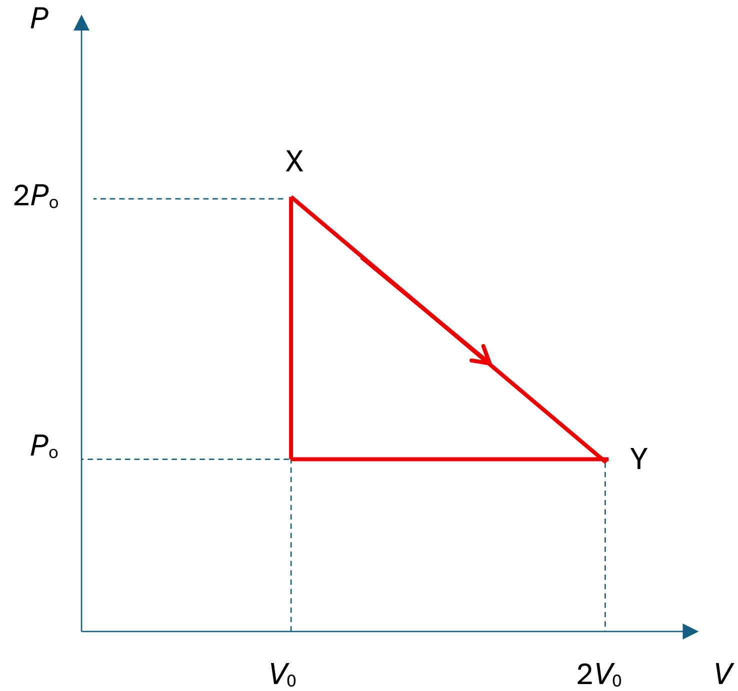
At $V = \frac{15}{8}$ we have $P = 3 - \frac{15}{8} = \frac{9}{8}$. The adiabatic through this point has equation $PV^{\gamma} = c = \frac{9}{8}(\frac{15}{8})^{\gamma}$. The gradient of the *P*-*V* line is – 1 and the gradient of the adiabatic is

$$\begin{aligned}\left.\frac{dP}{dV}\right|_{V=15/8} &= -\gamma\frac{9}{8}(\frac{15}{8})^{\gamma}V^{-\gamma-1}\\ &= -\gamma\frac{9}{8}(\frac{15}{8})^{\gamma}(\frac{15}{8})^{-\gamma-1}\\ &= -\frac{5}{3}\times\frac{9}{8}\times(\frac{15}{8})^{-1}\\ &= -1\end{aligned}$$

So, the adiabatic (red curve) is tangent to the *P*-*V* line:



So, the textbook problem asked for the efficiency of the cycle:



Along XY the heat in is $Q = \frac{49}{32}P_0V_0$. Heat enters the gas also along the vertical leg and that is

$$Q = \frac{3}{2}V\Delta P = \frac{3}{2}P_0V_0.$$

The total heat in is then $Q = \frac{49}{32}P_0V_0 + \frac{3}{2}P_0V_0 = \frac{97}{32}P_0V_0$.

The net work is the area of the loop i.e. $W_{net} = \frac{1}{2}P_0V_0$ and so the efficiency of the cycle is

$$e = \frac{\frac{1}{2}P_0V_0}{\frac{97}{32}P_0V_0} = \frac{16}{97} \approx 0.165.$$

This is slightly less than the $e = \frac{1}{6} \approx 0.167$ in the textbook.